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Toeplitz operators, deformations, and asymptotics

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Abstract

We consider deformations of commutators of Toeplitz operators and study norms and traces in the semi-classical limit.

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1. Introduction

Let (X, J, ω) be a compact connected n-dimensional Kähler manifold, $n \geq 1$. X denotes the underlying smooth manifold, and J, ω are the complex structure and the symplectic form respectively. Denote by $\{.,.\}$ the Poisson bracket. It gives a Lie algebra structure on the space $C^\infty(X)$ of all smooth real-valued functions on X. Assume that ω is integral. The standard set-up of Kähler quantization is as follows [3,9]. There is a holomorphic Hermitian line bundle $\mathcal{L} \to X$, with Hermitian connection ∇, such that $c_1(\mathcal{L}) = [\omega]$ and $\mathrm{curv}(\nabla) = -2\pi i\omega$. This line bundle is ample. Since $\mathcal{L}^{\otimes N_0}$ is very ample for sufficiently large N_0, we shall assume that $\mathcal{L}$ is very ample to begin with. This amounts, alternatively, to replacing ω by $N_0\omega$. Let N be a non-negative integer. Denote by d_N the dimension of the finite-dimensional complex vector space $\mathcal{H}_N = H^0(X, \mathcal{L}^{\otimes N})$. For any $f \in C^\infty(X)$ there is a Toeplitz operator

$$T_f = \bigoplus_{N=0}^{\infty} T_f^{(N)},$$

where $T_f^{(N)} = \Pi^{(N)} \circ M_f^{(N)} \in \mathrm{End}(\mathcal{H}_N)$,

$$M_f^{(N)} : H^0(X, \mathcal{L}^{\otimes N}) \to L^2(X, \mathcal{L}^{\otimes N})$$
$$s \mapsto fs,$$

is the operator of multiplication by f and

$$\Pi^{(N)} : L^2(X, \mathcal{L}^{\otimes N}) \to H^0(X, \mathcal{L}^{\otimes N})$$

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is the orthogonal projector. For every N the map

$$C^\infty(X) \to \mathrm{End}(\mathcal{H}_N)$$
$$f \mapsto T_f^{(N)} \tag{1}$$

is surjective (Proposition 4.2 [2]).

It is known that for a positive integer m, $f_1, \ldots, f_m \in C^\infty(X)$

$$tr(T_{f_1}^{(N)} \cdots T_{f_m}^{(N)}) = N^n \left(\int_X f_1 \cdots f_m \frac{\omega^n}{n!} + O\left(\frac{1}{N}\right)\right) \tag{2}$$

as $N \to +\infty$, see [2] Section 5, Remark 3, also [7] Section 4, proof of Lemma 5.

One of the main results of [2] (Theorem 4.2) states that for $f, g \in C^\infty(X)$

$$\|N[T_f^{(N)}, T_g^{(N)}] - \mathrm{i}T_{\{f,g\}}^{(N)}\| = O\left(\frac{1}{N}\right) \quad \text{as } N \to +\infty, \tag{3}$$

where $\| \cdot \|$ is the operator norm, i.e. $\|A^{(N)}\|^2 = \sup_{s\in(\mathcal{H}_N - 0)} \frac{\langle As, As\rangle_N}{\langle s,s\rangle_N}$ for $A^{(N)} \in \mathrm{End}(\mathcal{H}_N)$, and $\langle ., .\rangle_N$ is the Hermitian inner product on $\mathcal{H}_N$. Also

$$\|T_{fg}^{(N)} - T_f^{(N)} T_g^{(N)}\| = O\left(\frac{1}{N}\right) \quad \text{as } N \to +\infty \tag{4}$$

[2] p. 291 (2).

The equality

$$[T_f^{(N)}, T_g^{(N)}] = \frac{\mathrm{i}}{N} T_{\{f,g\}}^{(N)} \tag{5}$$

for all $f, g \in C^\infty(X)$, with N fixed, would give a finite-dimensional representation of the Lie algebra $C^\infty(X)$ (via $f \mapsto -\mathrm{i}NT_f^{(N)}$), however, it is known that $C^\infty(X)$ does not have non-trivial finite-dimensional representations [6]. The map (1) is linear, but it is neither a representation of $C^\infty(X)$ as a Lie algebra nor as associative algebra. It will only become an asymptotic representation in the semi-classical limit by (3) and (4).

Original motivation for this came from quantum mechanics, where the Poisson bracket and the commutator are two fundamental concepts. It is expected that the Poisson bracket of functions (classical observables) should "correspond" to the commutator of operators (quantum observables), which means that one would like to have a representation of $C^\infty(X)$ in the Hilbert space that consists of quantum-mechanical states (wave functions). Here the Hilbert space is $\mathcal{H}_N$ and the Planck constant, philosophically, is $\hbar = \frac{1}{N}$.

In this note, first, we show that a statement similar to (3) holds if one deforms the commutator in an N-dependent way, varying the Lie algebra structure on $\mathrm{End}(\mathcal{H}_N)$ in the variety of d_N^2-dimensional complex Lie algebras sufficiently close to $gl(d_N, \mathbb{C})$, this is done in Section 2. Then, in Section 3, we study certain q-deformations of the commutator (independent of N), prove a result which is similar in spirit to (3), and define the corresponding q-deformation of the Poisson bracket.

The main results of the paper are Theorems 2.1 and 3.1.

2. Varying Lie algebra structure

Fix a positive integer r. Pick a basis $X_1, \ldots, X_r$ of $\mathbb{C}^r$. Consider the variety $\mathcal{A}_r \subset \mathbb{C}^{r^3}$ that parametrizes the r-dimensional complex Lie algebra structures, i.e.

$$\mathcal{A}_r = \{\{c_{mj}^k\}, m, j, k = 1, \ldots, r | c_{mj}^k = -c_{jm}^k, c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m = 0\}.$$

Here c_{mj}^k are the structure constants, i.e. $[X_m, X_j] = c_{mj}^k X_k$, and the two sets of conditions are, respectively, anti-symmetry and the Jacobi identity. We follow the standard conventions for tensor notations, in particular we omit the summation sign.

The structure of $\mathcal{A}_r$ and of the orbits of the natural $GL(r, \mathbb{C})$-action on $\mathcal{A}_r$ is thoroughly investigated in [8,10], see also [15].

For $c = \{c^k_{mj}\} \in \mathcal{A}_r$, we shall denote by $\mathcal{B}_r(c, R)$ the open ball in $\mathbb{C}^{r^3}$ with center c and of radius R with respect to the standard Riemannian metric on $\mathbb{C}^{r^3}$.

The equality (3) gives an estimate of how close the map (1) is to being a Lie algebra homomorphism asymptotically as $N \to +\infty$. The Lie algebra structure on $\mathrm{End}(\mathcal{H}_N)$ is the standard Lie algebra structure of $gl(d_N, \mathbb{C})$.

Denote $r = r(N) = d_N^2 = \dim \mathrm{End}(\mathcal{H}_N)$. For every $N \geq 1$ pick an orthonormal basis $\{s_j^{(N)}\}$, $j = 1, \dots, d_N$, in $H^0(X, \mathcal{L}^{\otimes N})$.

For $p, q = 1, \dots, r$ define $E^{(N)}_{p,q}$ to be the $r \times r$ matrix with the (p, q)th entry equal to 1 and all other entries equal to zero. Define the basis $X_1^{(N)}, \dots, X_{r(N)}^{(N)}$ in $gl(d_N, \mathbb{C})$ as follows: $E^{(N)}_{p,q} = X^{(N)}_{(p-1)d_N+q}$.

Denote by $\tilde{c}(N) = \{\tilde{c}^k_{mj}(N)\} \in \mathcal{A}_{r(N)}$ the point in $\mathcal{A}_{r(N)}$ corresponding to $gl(d_N, \mathbb{C})$. For any $c = c(N) = \{c^k_{mj}(N)\} \in \mathcal{A}_{r(N)}$ we shall denote by $[.,.]_{c(N)}$ the corresponding Lie bracket on $\mathrm{End}(\mathcal{H}_N)$. So $[.,.]_{\tilde{c}(N)} = [.,.]$.

Varying the Lie algebra structure on $\mathrm{End}(\mathcal{H}_N)$ leads to the following generalization of (3):

Theorem 2.1. *Let $f, g \in C^\infty(X)$. Let a be a positive constant. For all sequences $\{c(N)\}_{N=1}^{+\infty}$ such that $c(N) \in \mathcal{A}_{r(N)} \cap \mathcal{B}_{r(N)}(\tilde{c}(N), \epsilon(N))$, where $\epsilon(N) = aN^{-6n-2}$,*

(i) $$\|N[T_f^{(N)}, T_g^{(N)}]_{c(N)} - \mathrm{i}T^{(N)}_{\{f,g\}}\| = O\left(\frac{1}{N}\right) \quad \text{as } N \to +\infty;$$

(ii) $$\frac{1}{d_N} tr[T_f^{(N)}, T_g^{(N)}]_{c(N)} = O\left(\frac{1}{N}\right) \quad \text{as } N \to +\infty.$$

Corollary 2.1. $\|[T_f^{(N)}, T_g^{(N)}]_{c(N)}\| = O(\frac{1}{N})$ *as* $N \to +\infty$.

Remark 2.1. The asymptotics in Theorem 2.1 and Corollary 2.1 do not depend on the choice of the orthonormal bases $\{s_j^{(N)}\}$.

Remark 2.2. The authors of [2] point out that the maps (1) provide an approximation of the Poisson algebra $(C^\infty(X), \{.,.\})$ by a sequence of finite-dimensional Lie algebras isomorphic to $gl(d_N, \mathbb{C})$, $N \to +\infty$. Theorem 2.1 provides other approximations of $(C^\infty(X), \{.,.\})$ by sequences of finite-dimensional complex Lie algebras. These Lie algebras are also isomorphic to $gl(d_N, \mathbb{C})$, this follows from [10], Lemma on p. 308, and from the fact that $sl(d_N, \mathbb{C})$ is simple, and hence does not have non-trivial deformations.

Remark 2.3. $\oplus_{N=0}^{+\infty}[T_f^{(N)}, T_g^{(N)}]_{c(N)}$ is not necessarily a Toeplitz operator.

Remark 2.4. The equality

$$\lim_{N\to+\infty} \|[T_f^{(N)}, T_g^{(N)}]_{c(N)}\| = 0$$

can be interpreted as “quantum observables commute in the semi-classical limit”.

Proof of Theorem 2.1. Proof of (i).

We have: $T_f^{(N)} = \alpha^m(N)X_m^{(N)}$, $T_g^{(N)} = \beta^j(N)X_j^{(N)}$,

$$[T_f^{(N)}, T_g^{(N)}]_{c(N)} = \alpha^m(N)\beta^j(N)c^k_{mj}(N)X_k^{(N)}.$$

By Theorem 4.1 [2]

$$\|T_f^{(N)}\| \leq \|f\|_\infty, \qquad \|T_g^{(N)}\| \leq \|g\|_\infty,$$

for large N, where $\|f\|_\infty$ denotes the sup-norm of f on X. Also

$$\|T_f^{(N)}\|^2 \geq \langle T_f^{(N)}s_j^{(N)}, T_f^{(N)}s_j^{(N)}\rangle_N,$$

hence

$$\left|\max_{1\le m\le r(N)} \alpha^m(N)\right| \le \|f\|_\infty, \qquad \left|\max_{1\le j\le r(N)} \beta^j(N)\right| \le \|g\|_\infty$$

for sufficiently large N. We also note that $d_N \sim \frac{1}{n!}\mathrm{vol}_{\omega^n}(X)N^n$ as $N \to +\infty$, hence there is a constant $K > 0$ such that $r(N) \le KN^{2n}$ as $N \to +\infty$. Therefore for sufficiently large N

$$\begin{aligned}\|[T_f^{(N)}, T_g^{(N)}]_{c(N)} - [T_f^{(N)}, T_g^{(N)}]_{\tilde{c}(N)}\| &= \|\alpha^m(N)\beta^j(N)(c^k_{mj}(N) - \tilde{c}^k_{mj}(N))X_k^{(N)}\| \\ &\le K^3N^{6n}\|f\|_\infty\|g\|_\infty\epsilon(N) = \frac{aK^3\|f\|_\infty\|g\|_\infty}{N^2}. \end{aligned} \tag{6}$$

Then

$$\begin{aligned}\|N^2[T_f^{(N)}, T_g^{(N)}]_{c(N)} - \mathrm{i}NT^{(N)}_{\{f,g\}}\| &\le \|N^2[T_f^{(N)}, T_g^{(N)}]_{c(N)} - N^2[T_f^{(N)}, T_g^{(N)}]_{\tilde{c}(N)}\| \\ &\quad + \|N^2[T_f^{(N)}, T_g^{(N)}]_{\tilde{c}(N)} - \mathrm{i}NT^{(N)}_{\{f,g\}}\|,\end{aligned}$$

(6) implies that the first term is bounded, and the second term is bounded because of (3).

Proof of (ii).

$$tr[T_f^{(N)}, T_g^{(N)}]_{c(N)} = \alpha^m(N)\beta^j(N)(c^k_{mj}(N) - \tilde{c}^k_{mj}(N))tr(X_k^{(N)}) + tr[T_f^{(N)}, T_g^{(N)}],$$

the first term is $O(\frac{1}{N^2})$ and by (2) the second term is $O(N^{n-1})$, so we get $O(N^{n-1})$. □

Proof of Corollary 2.1.

$$N\|[T_f^{(N)}, T_g^{(N)}]_{c(N)}\| \le \|N[T_f^{(N)}, T_g^{(N)}]_{c(N)} - \mathrm{i}T^{(N)}_{\{f,g\}}\| + \|T^{(N)}_{\{f,g\}}\|,$$

the first term is $O(\frac{1}{N})$ by Theorem 2.1(i) and the second term is bounded by $\|\{f, g\}\|_\infty$ (Theorem 4.1 [2]). □

3. q-deformations of commutator

For a non-negative integer N, $q \in \mathbb{R}$, $a, b \in C^\infty(\mathbb{R})$ such that $a(1) = b(1) = 1$, let us define a bilinear map

$$\begin{aligned}&\mathrm{End}(\mathcal{H}_N) \times \mathrm{End}(\mathcal{H}_N) \to \mathrm{End}(\mathcal{H}_N)\\ &X, Y \mapsto [X, Y]_q,\end{aligned}$$

where

$$[X, Y]_q = a(q)XY - b(q)YX. \tag{7}$$

Thus for $f, g \in C^\infty(X)$

$$[T_f, T_g]_q = a(q)T_fT_g - b(q)T_gT_f.$$

It is clear that the condition of antisymmetry and Jacobi identity are satisfied if and only if $a(q) = b(q)$, i.e. otherwise we do not get a Lie bracket.

Note that if $a(q) \ne b(q)$ then by (2)

$$tr[T_f^{(N)}, T_g^{(N)}]_q \sim N^n(a(q) - b(q)) \int_X fg\frac{\omega^n}{n!}$$

as $N \to +\infty$ and if $a(q) = b(q)$ then $tr[T_f^{(N)}, T_g^{(N)}]_q = O(N^{n-1})$.

Proposition 3.1. *Suppose $q \in \mathbb{R}$, $a, b \in C^\infty(\mathbb{R})$ are such that $a(q) = b(q) \ne 0$. Then for any $f, g \in C^\infty(X)$*

(i) $\|N[T_f^{(N)}, T_g^{(N)}]_q - \mathrm{i}T^{(N)}_{a(q)\{f,g\}}\| = O\left(\frac{1}{N}\right)$ *as* $N \to +\infty$;

(ii) $\|[T_f^{(N)}, T_g^{(N)}]_q\| = O(\frac{1}{N})$ *as* $N \to +\infty$.

The proof of (i) goes exactly as the proof of Theorem 4.2 in [2], and (ii) follows immediately.

Theorem 3.1. *Suppose $a, b \in C^\infty(\mathbb{R})$, $a(1) = b(1) = 1$.*

(i) *If $q \in \mathbb{R}$ is such that $a(q) \neq b(q)$, then*

$$\|[T_f^{(N)}, T_g^{(N)}]_q - T_{(a(q)-b(q))fg}^{(N)}\| = O\left(\frac{1}{N}\right) \quad \text{as } N \to +\infty.$$

(ii) *For each $q \in \mathbb{R}$ there is a bilinear map*

$$C^\infty(X) \times C^\infty(X) \to C^\infty(X)$$
$$f, g \mapsto \{f, g\}_q$$

such that $\{f, g\}_1 = \{f, g\}$, and for fixed $q \in \mathbb{R}$

$$\|N([T_f^{(N)}, T_g^{(N)}]_q - T_{(a(q)-b(q))fg}^{(N)}) - \mathrm{i}T_{\{f,g\}_q}^{(N)}\| = O\left(\frac{1}{N}\right) \quad \text{as } N \to +\infty.$$

Corollary 3.1. *If $a(q) \neq b(q)$ then $\|[T_f^{(N)}, T_g^{(N)}]_q\| = O(1)$ as $N \to +\infty$.*

Remark 3.1. If $a(q) \neq b(q)$ then $\{.,.\}_q$ is not a Poisson bracket.

Proof of Theorem 3.1. The proofs relies on ideas of [4], see also [2] Section 5, and [3] Section 13.

Denote by P the unit circle bundle in $\mathcal{L}^*$, it is the boundary of a strictly pseudoconvex domain (the disc bundle). Denote by α the contact form on P. Note that $\oplus_{N=0}^{+\infty} \mathcal{H}_N \subset L^2(P)$. Consider a symplectic submanifold $\Sigma = \{(p, r\alpha_p) | p \in P, r > 0\} \subset T^*P - \{0\}$, of T^*P. The generator of the circle action on the fiber of P gives a first order Toeplitz operator D_φ. D_φ acts on $\mathcal{H}_N$ by multiplication by N. Let $\tau : \Sigma \to X, \pi : P \to X$ be the natural projections. We have: $\{\tau_\Sigma^* f, \tau_\Sigma^* g\}_\Sigma(p, r\alpha_p) = \frac{1}{r}\{f, g\}(\pi(p))$.

Proof of (i). $[T_f, T_g]_q$ is a Toeplitz operator of order zero with principal symbol $(a(q) - b(q))fg$. Hence $[T_f, T_g]_q - T_{(a(q)-b(q))fg}$ is a Toeplitz operator of order -1. Then $A := D_\varphi([T_f, T_g]_q - T_{(a(q)-b(q))fg})$ is a Toeplitz operator of order zero, and, since X is compact, it is bounded (see the argument in [2], Section 5). We have: $A = \oplus_{N=0}^{+\infty} A^{(N)}$, where

$$A^{(N)} = A|_{\mathcal{H}_N} = N([T_f^{(N)}, T_g^{(N)}]_q - T_{(a(q)-b(q))fg}^{(N)}),$$

$\|A^{(N)}\| \leq \|A\|$, and (i) follows.

Proof of (ii). If $a(q) = b(q)$ then we are in the situation of Proposition 3.1, in particular $\{f, g\}_q = a(q)\{f, g\}$.

Now suppose that $a(q) \neq b(q)$. Define $\{f, g\}_q$ to be $\frac{1}{i}$ times the principal symbol of A. Then the first order Toeplitz operator $B := D_\varphi(A - \mathrm{i}T_{\{f,g\}_q})$ has vanishing principal symbol, so it is in fact of order zero, and therefore it is bounded. We have: $B = \oplus_{N=0}^{+\infty} B^{(N)}$, where $B^{(N)} = B|_{\mathcal{H}_N} = N(A^{(N)} - \mathrm{i}T_{\{f,g\}_q}^{(N)})$, also $\|B^{(N)}\| \leq \|B\|$, and we are done. □

Proof of Corollary 3.1.

$$\|[T_f^{(N)}, T_g^{(N)}]_q\| \leq \|[T_f^{(N)}, T_g^{(N)}]_q - T_{(a(q)-b(q))fg}^{(N)}\| + \|T_{(a(q)-b(q))fg}^{(N)}\|,$$

the first term is $O(\frac{1}{N})$ by Theorem 3.1(i), and the second term is bounded by $|(a(q) - b(q))| \, \|fg\|_\infty$ (Theorem 4.1 [2]). □

Remark 3.2. The deformation $\{.,.\}_q$ is defined essentially as the coefficient at the second term in the asymptotic expansion of the symbol of $[T_f, T_g]_q$, up to the factor of $\frac{1}{i}$ (see the proof of Theorem 3.1(ii)). As usual, we write the symbol of composition of operators and take the linear combination with coefficients $a(q), -b(q)$. If $a(q) = b(q)$, then the first terms cancel and the second terms give the usual Poisson bracket, times $\mathrm{i}a(q)$. Otherwise we get $\{.,.\}_q$ as described above.

Remark 3.3. Here is an explicit example of $\{.,.\}_q$. Let X be a $2n$-dimensional torus, $n \geq 1$, with the standard symplectic form $\omega = \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j$ and with the standard complex structure. Then

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x_j} \right)$$

and

$$\{f, g\}_q = \sum_{j=1}^{n} \left(a(q) \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} - b(q) \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x_j} \right). \tag{8}$$

Remark 3.4. Formula (8) is, in fact, for $\mathbb{R}^{2n}$ and it gives an expression of the deformation on arbitrary X in local (Darboux) coordinates.

Remark 3.5. We note that for $f, g \in C^{\infty}(X)$

$$\mathrm{i}\{f, g\}_q = a(q) C_1(f, g) - b(q) C_1(g, f),$$

where $C_1(f, g)$ is the second coefficient of the Berezin–Toeplitz star product [11]. Also the main result of [11] (Theorem 2.2) implies that

$$\left\| T_f^{(N)} T_g^{(N)} - \left(T_{fg}^{(N)} + \frac{1}{N} T_{C_1(f,g)}^{(N)} \right) \right\| = O\left(\frac{1}{N^2} \right),$$

and this gives another proof of Theorem 3.1.

Remark 3.6. Here is an informal comment explaining a motivation for the choice of deformation (7). q-deformed commutators appear in the physical literature, in particular, in the following context. Recall that the three-dimensional Heisenberg algebra can be realized as the algebra (over $\mathbb{C}$) generated by operators Q, P, I on $L^2(\mathbb{R}, \mathrm{d}x)$, where $Qf(x) = xf(x)$, $Pf(x) = -\mathrm{i}\hbar \frac{\mathrm{d}f}{\mathrm{d}x}$ and I is the identity operator. Historically, the commutator appeared in quantum mechanics as the Lie bracket on the Heisenberg algebra. In particular, $[Q, P] = QP - PQ = \mathrm{i}\hbar I$. It comes from the usual commutator on the ring of pseudodifferential operators on $\mathbb{R}^2$.

The q-deformed Heisenberg algebra is, by definition, an algebra over $\mathbb{C}$, generated by elements $1 \in \mathbb{C}$, A, B, with relation $AB - qBA = p$, where $p, q \in \mathbb{C}$ [14]. Let us assume that q is real and $p = \mathrm{i}\hbar$. The corresponding q-deformed quantum-mechanical object is referred to as "q-deformed harmonic oscillator" or "q–$\hbar$-deformed system" and it has been extensively studied in the physical literature. See, in particular, [1,5,12,13], and references in these papers. And

$$A, B \mapsto AB - qBA \tag{9}$$

is the corresponding "q-deformed commutator" on the ring of pseudodifferential operators on $\mathbb{R}^2$.

Expression (7), with $a(q) = 1$, $b(q) = q$, is, formally, the q-deformed commutator as above. More accurately, one should say that, first, we define the q-deformed commutator on the ring of pseudodifferential operators on $\mathbb{R}^{2n}$, again, by (9), then we note that the picture on an open subset of $\mathbb{R}^{2n}$ provides a local picture in a Darboux coordinate chart on the symplectic manifold. And we define the deformed commutator on the ring of Toeplitz operators by (9) too.

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